SOLUTIONS (TEST SERIES 4) PH

1. (3) We know vector grad ϕ is perpendicular to level surface

$\therefore$ Given surface $z = x^2 + y^2$

or $x^2 + y^2 - z = 0$

normal to this surface

$$= \nabla\left(x^2 + y^2 - z\right)$$

$$= \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 - z\right)$$

$$= (2x)\hat{i} + (2y)\hat{j} + (-1)\hat{k}$$

normal at the point (1,2,5) is given by

$$= (2\times1)\hat{i} + (2\times2)\hat{j} + (-1)\hat{k}$$

$$= 2\hat{i} + 4\hat{j} - \hat{k}$$

Unit vector normal to the surface

$$= \pm\frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{(2)^2 + (4)^2 + (-1)^2}}$$

$$\boxed{= \pm\frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}}$$

2. (3) Performing the operation

$R_1 \to \frac{1}{2}R_1, R_2 \to \frac{1}{2}R_2$, we have

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix}$$

by $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} \quad \text{by } \begin{array}{l} C_2 \to C_2 + C_1 \\ C_4 \to C_4 - 3C_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \end{bmatrix} \quad \text{by } R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -5 \end{bmatrix} \quad \text{by } R_2 \to (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix} \quad \text{by } R_4 \to R_4 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -8 \end{bmatrix} \quad \text{by } \begin{array}{l} C_3 \to C_3 + C_2 \\ C_4 \to C_4 - C_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{by } R_4 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{by } \begin{array}{l} C_3 \to \frac{1}{3}C_3 \\ C_4 \to \frac{1}{8}C_4 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{by } C_4 \to C_4 - C_3$$

Which is the normal for $\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $\boxed{rank\ A = 3}$

3. (1)

4. (3) The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

or $(m-1)(m-2) = 0$

$\therefore$ $m = 1, 2$

Hence $CF = c_1e^x + c_2e^{2x}$

and $$P.I. = \frac{1}{D^2 - 3D + 2} \cosh x$$

$$= \frac{1}{D^2 - 3D + 2}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^2 - 3D + 2}e^x + \frac{1}{2}\frac{1}{D^2 - 3D + 2}e^x$$

$$= \frac{1}{2}\frac{1}{(D-1)(D-2)}e^x + \frac{1}{2}\frac{1}{(-1)^2 + 3 + 2e^{-x}}$$

$$= -\frac{1}{2}\frac{1}{D-1}e^x.1 + \frac{1}{12}e^{-x}$$

$$= -\frac{1}{2}e^x \frac{1}{D+1-1}1 + \frac{1}{12}e^{-x}$$

$$= -\frac{1}{2}e^x \frac{1}{D}1 + \frac{1}{12}e^{-x}$$

$$= -\frac{1}{2}xe^x + \frac{1}{12}e^{-x}$$

general solution

$$\boxed{y = c_1e^x + c_2e^{2x} - \frac{1}{2}xe^x + \frac{1}{12}e^{-x}}$$

5. (1)

6. (3) $|z-3| + |z+3| = 10$

$|x+iy-3| + |x+iy+3| = 10$

$$\sqrt{(x-3)^2 + y^3} + \sqrt{(x+3)^2 + y^2} = 10$$

$$(x-3)^2 + y^2 = 100 + (x+3)^2 + y^2 - 20\sqrt{(x+3)^2 + y^2}$$

$$x^2 - 6x + 9 + y^2 - 100 - x^2 - 6x - 9 - y^2 = -20\sqrt{(x-3)^2 + y^2}$$

$$100 + 12x = 20\sqrt{(x+3)^2 + y^2}$$

$$(x+3)^2 + y^2 = \left(\frac{100+12x}{20}\right)^2$$

$$25(x^2 + 6x + 9 + y^2) = (25)^2 + 2 \times 25 \times 3x + 9x^2$$

$$25x^2 + 150x + 225 + 25y^2 = 625 + 150x + 9x^2$$

$$16x^2 + 25y^2 = 400$$

$$\frac{16x^2}{400} + \frac{25y^2}{400} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

7. (2) using result of previous questions

$a = 5$ and $b = 4$

$$\therefore \quad I = \frac{2\pi.5}{\sqrt{(25-16)^3}} = \frac{10\pi}{27}$$

$$\Rightarrow \quad I = \frac{10\pi}{27}$$

8. (1)
9. (1)

10. (1) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

$\lambda_1, \lambda_2, \lambda_3 \Rightarrow 1, -2, -3$

Then for

$A^3 \Rightarrow 1, -8, -27$

$A^2 \Rightarrow 1, 4, 9$

$A \Rightarrow 1, -2, -3$

$I \Rightarrow 1, 1, 1$

The eigen value for $3A^3 + 5A^2 + 6A + I$ will be

1. $3(1) + 5(1) + 6(1) + 1 = 5$
2. $3(-8) + 5(4) + 6(-2) + 1 = -24 + 20 - 12 + 1 = -15$
3. $3(-27) + 5(9) + 6(-3) + 1 = -53$

11. (3)
12. (1)
13. (1)
14. (1)

15. (2) here

$$u_n = \frac{n+1}{n^p} = \frac{1 + \frac{1}{n}}{n^{p-1}}$$

Let $v_n = n^{p-1}$

$$\therefore \quad \frac{u_n}{v_n} = 1 + \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = 1$$

$\therefore$ both the series are either convergent or divergent but Σu_n is convergent if $p - 1 > 1$ i.e., if $p > 2$ and is divergent if $p - 1 \le 1$ i.e. if $p \le 2$

Therefore, the given series is convergent if $p > 2$ and divergent if $p \le 2$

16. (2) given series $\left(n^{\log x}\right)$ can be writing as

$$=\frac{1}{1^{-\log x}}+\frac{1}{2^{-\log x}}+\frac{1}{3^{-\log x}}+....$$

Substitute $-\log x = p$

$$=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+...\infty$$

If $p>1$, i.e., $-\log x>1$

$$\Rightarrow \quad \log_e \frac{1}{x}>\log_e e$$

or $\frac{1}{x}>e$

or $x<\frac{1}{e}$

Series is convergent

If $p\le 1 \Rightarrow -\log x \le 1$

$$\Rightarrow \quad \log_e \frac{1}{x}\le \log_e e$$

or $\frac{1}{x}\le e$

or $\frac{1}{e}\le x$

series is divergent

17. (2) given series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+.....\infty$

$$|u_n|=\frac{1}{n^2}$$

$$|u_{n+1}|=\frac{1}{n^2}$$

(i) $|u_{n+1}|<|u_n|$

(ii) $\lim\limits_{n\to\infty} u_n=\lim\limits_{n\to\infty}\frac{1}{n^2}=0$

Series is convergent by Leibnitz's rule

and $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+....$ is also

Convergent (p series, $p=2$)

Thus series is absolutely convergent

18. (2)

19. (1) $X(j\omega)=\int\limits_{-\infty}^{\infty} e^{4|t|}e^{-j\omega t}dt$

$$=\int\limits_{-\infty}^{0} e^{-4t}e^{-j\omega t}dt+\int\limits_{0}^{\infty} e^{-4t}e^{-j\omega t}dt$$

$$=\frac{8}{16+\omega^2}$$

20. (2) $e^{z0}=-3$

or $e^z=3(-1)$

$$=3\left[e^{i(2n+1)\pi}\right]$$

with $n=0,\pm1,\pm2,......$

taking log on both side

$$\log(e^z)=\log\left[3(e^{i(2n+1)\pi})\right]$$

$$z=3+i(2n+1)\pi$$

with $n=0,\pm1,\pm2....$

21. (1)
22. (4)
23. (3)
24. (1)
25. (2)
26. (3)
27. (4)

28. (4) The probability of drawing a red ball

$$=\frac{5}{10}$$

If the ball is not replaced, the box will have a ball, so probability of drawing the red ball in next chance $=\frac{4}{9}$

Hence the probability of drawing 2 balls

$$=\frac{5}{10}\times\frac{4}{9}=\frac{2}{9}$$

29. (1)
30. (1)
31. (2)
32. (4)
33. (3)
34. (1)
35. (1)
36. (3)
37. (1)
38. (1)
39. (1)
40. (1)
41. (4)
42. (4)
43. (2)
44. (3)

45. (1) Given

$n_{eff}=4, M=108\text{kg},$

$\rho=3.32\text{gm cm}^{-3}=3320\text{kgm}^{-3}$

$N_A=6.023\times10^{+26}$ atoms | kmd

$$a^3 = \frac{n_e \times M}{N_A \times \rho} = \frac{4\times 108}{6.023\times 10^{26}\times 3320} = 6.00\times 10^{-30} m^3$$
$$= 6.00\times 10^{-10} = 6.00 A^o$$

46. (2) According to Bragg's law

$$2d\sin\theta = \lambda, \sin\theta = \frac{\lambda}{2d}$$

Where

$$d = \frac{q}{\sqrt{h^2+k^2+l^2}} = \frac{a}{\sqrt{3}} \text{ for (111)plane}$$

$\therefore$

$$\sin\theta = \frac{\sqrt{3}\lambda}{2a} = \frac{\sqrt{3}\times 1.5A^o}{2\times 6A^o} = \frac{\sqrt{3}\times 3}{2\times 62} = \frac{\sqrt{3}}{8}$$

47. (1) Given fluxoid

$$(\phi)_0 = 2\times 10^{-7}\ \text{gauss} - \text{cm}^2$$

First critical field $(H_{c1}) = \frac{2}{\pi}\times 10^5$ gauss

The relation between first critical field and penetration depth is

$$H_{c1} = \frac{\phi_0}{\pi\lambda^2} \therefore \lambda^2 = \frac{\phi_0}{\pi H_{c1}} = \frac{2.10^{-7}}{\pi\times\frac{2}{\pi}\times 10^5}$$

$$= 10^{-12} cm^2 \Rightarrow \lambda = 10^6 cm = 100 A^o$$

48. (4)

Given $V(r) = -\frac{a}{r^6} + \frac{b}{r^{12}}$

At equilibrium radius, $\left.\frac{dV(r)}{dr}\right|_{r=r_0} = 0$

$$\therefore \frac{dV(r)}{dr} = +\frac{6a}{r_0^7} - \frac{12b}{r_0^{13}} = 0$$

$$\Rightarrow \frac{r_0^{13}}{r_0^7} = \frac{12b}{6a} = \frac{2b}{a} \Rightarrow r_0^6 = \frac{2b}{a}$$

$\therefore$The value of potential at equilibrium is

$$V(r_0) = -\frac{a}{r_0^6} + \frac{b}{r_0^{12}} = -\frac{a^2}{2b} + \frac{a^2}{4b} = -\frac{a^2}{4b}$$

49. (4)

For large $\lambda, (k_x a, k_y a, k_z a)$ are small

$$\omega^2(k) = \omega_0^2\left[3 - \left(1 - \frac{k_x^2 a^2}{2}\right) - \left(1 - \frac{k_y^2 a}{2}\right) - \left(1 - \frac{k_z^2 a^2}{2}\right)\right]$$

$$= \frac{\omega_0^2 a^2}{2}(k_x^2 + k_y^2 + k_z^2)$$

$$\omega^2(k) = \frac{\omega_0^2 a^2}{2} k^2 \Rightarrow \omega = \frac{\omega_0 a}{\sqrt{2}} k$$

$$\Rightarrow v_g = \frac{d\omega}{dk} = \frac{\omega_0 a}{\sqrt{2}}$$

50. (3)