SOLUTIONS (TEST SERIES 5) PH

1. (2)
2. (3)
3. (4)
4. (4)
5. (4)
6. (1)
7. (1)
8. (4)
9. (2)
10. (1)
11. (3)
12. (1)
13. (2)
14. (1)
15. (1)
16. (1)
17. (3)
18. (3)
19. (1)
20. (3)

21. $\mathrm{I} = \int_{-1}^{1} (1+2n) P_n(x)\,dx$

$$= \int_{-1}^{1} P_0(x).P_n(x)\,dx + 2\int_{-1}^{1} P_1(x)\,P_n(x)\,dx$$

Use $\int_{-1}^{1} P_n(x).P_m(x)\,dx = \frac{2}{2n+1}\delta_{m,n}$

$\therefore \quad \mathrm{I} = \frac{2}{2n+1}\delta_{n,0} + \left(\frac{2}{2n+1}\right)\delta_{n,1}$

Since $n>1,\ \delta_{n,0}=0,\ \delta_{n,1}=0$

$\mathrm{I}=0$

22. $\frac{d^2y}{dx^2} = y$

$\frac{d^2y}{dx^2} - y = 0$

$(D^2-1)\,y=0$

$D = \pm 1$

$\therefore \quad y = C_1 e^{x} + C_2 e^{-x}$

$y(0)=0 \Rightarrow C_1 + C_2 = 0 \ldots.\ (1)$

$y(\infty)=0 \Rightarrow C_1 + 0 = 0$

$\boxed{C_1 = 0}$

Now from equation (1)

$0 + C_2 = 0$

$\boxed{C_2 = 0}$

means $C_1 = C_2 = 0$

Then has no solution

23. $S = 2+1+\frac{2}{3}+\frac{1}{2}+\frac{2}{5}+\frac{1}{3}+\frac{2}{7}+\frac{1}{4}$

$$= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+2\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)$$

Now $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$\log(1-x) = -x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$$\therefore \log(1+x) - \log(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{4}\right]$$

$$\therefore S = \lim_{x\to 1}\left[-\log(1-x) + \log(1+x) - \log(1-x)\right]$$

$$= \lim_{x\to 1}\left[-2\log(1-x) + \log(1+x)\right]$$

$$S = \lim_{x\to 1} \log\left[\frac{1+x}{(1-x)^2}\right]$$

$\boxed{S = \infty}$

So, S diverges to ∞

24. $P^2 = P$

$P^2 - P = 0$

$P(P-1) = 0$

$P = 0,1$ (eigen values)

Since, $T_r(P) = n-1$ and trace is equal to sum of eigen values. So, one of eigen value will be zero

Hence Det(P) = product of eigen values = 0

25. (3)

26. wave function for ground state of H-like atom

$$\Psi = \sqrt{\frac{z^3}{\pi a_0^3}}\, e^{-zr/a_0}$$

Initial state $\Psi_i =$ ground state of ${}_1^3\mathrm{H}$ atom

$z = 1$

$$= \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}$$

Final state $\Psi_f =$ ground state of He^+ ion

$(z=2)$

Required probability $P = \langle \Psi_f \mid \Psi_i \rangle^2$

Now

$$\langle \Psi_f | \Psi_i \rangle = \int_0^\infty \Psi_f^* \Psi_i^* d\tau = \int_0^\infty \frac{1}{\sqrt{\pi a_0^3}} e^{-3r/a_0} 4\pi r^2 dr$$

$$= \frac{8\sqrt{2}}{a_0^3} \int_0^\infty e^{-3r/a_0} r^2 \, dr$$

$$= \frac{8\sqrt{2}}{a_0^3} \times \frac{2!}{(3/a_0)^3} = \frac{16\sqrt{2}}{27}$$

$$\therefore \text{Probability} = \left| \frac{16\sqrt{2}}{27} \right|^2 = \frac{512}{729}$$

27. $$E_0^{'} = \left(\frac{2}{a}\right) \int_{-a/2}^{a/2} \cos^2 \frac{\pi x}{a} 2 \in \left|\frac{x}{a}\right| dx$$

$$= \frac{2}{a} x^2 \int_0^{a/2} \cos^2 \frac{\pi x}{a} 2 \in \frac{x}{a} dx$$

$$= \frac{8\in}{a^2} \int_0^{a/2} x \cos^2 \frac{\pi x}{a} dx$$

$$= \frac{8\in}{a^2} \left[\int_0^{a/2} x \frac{\left(1 + \cos \frac{2\pi x}{a}\right)}{2} \right] dx$$

$$= \frac{4\in}{a^2} \left[\int_0^{a/2} x \, dx + \int_0^{a/2} x \cos \frac{2\pi x}{a} dx \right]$$

$$= \frac{4\in}{a^2} \left[\left(\frac{x^2}{2}\right)_0^{a/2} + \frac{a}{2\pi} x \sin \frac{2\pi x}{a}\Big|_0^{a/2} - \int_0^{a/2} \frac{d}{dx} x \cdot \frac{a}{2\pi} \sin \frac{2\pi x}{a} dx \right]$$

$$= \frac{4\pi}{a^2} \left[\frac{a^2}{8} + 0 + \left(\frac{a}{2\pi}\right)^2 \cos \frac{2\pi x}{a}\Big|_0^{a/2} \right]$$

$$= \frac{4\in}{a^2} \left[\frac{a^2}{8} + \left(\frac{a}{2\pi}\right)^2 (-1-0) \right]$$

$$= \frac{4\in}{a^2} \left[\frac{a^2}{8} - 2\left(\frac{a}{2\pi}\right)^2 \right]$$

$$= \frac{4\in}{a^2} \left[\frac{a^2}{8} - \frac{2a^2}{4\pi} \right] = 4\in \left[\frac{1}{8} - \frac{2}{4\pi^2} \right]$$

$$= \frac{4\in}{8\pi^2} \left[\pi^2 - 4 \right]$$

$$\boxed{E_0^1 = \frac{\in}{2\pi^2} \left[\pi^2 - 4 \right]}$$

28. $$|\Psi(r,0)\rangle = \frac{1}{\sqrt{2}} \left[|1\rangle + |2\rangle \right]$$

$$|\Psi(r,t)\rangle = \frac{1}{\sqrt{2}} \left[|1\rangle e^{-iE_1T/\hbar} + |2\rangle e^{-iE_2T/\hbar} \right]$$

$$= e^{-iE_1T/\hbar} \left[|1\rangle + |2\rangle e^{-i(E_2-E_1)T/\hbar} \right]$$

$$= e^{-i(E_2 - E_1T/\hbar)} = -1$$

$$= e^{-i(E_2-E_1)T/\hbar} = e^{-\pi i}$$

$$= (E_2 - E_1)\frac{T}{\hbar} = gp$$

$$= E_2 - E_1 = \frac{\pi\hbar}{T}$$

$$= \frac{\pi}{T} \cdot \frac{h}{2\pi} \Rightarrow \boxed{E_2 - E_1 = \frac{h}{2T}}$$

29. $\Psi(x_1x_2) = \Psi_1(x_1)\Psi_2(x_2) \rightarrow$ symmetric space part

$S_1 = 1,\ S_2 = 1$

$S = 0, 1, 2$

$P = (-1)^S$

For total symmetric wave function, the spin part should be symmetric

$\therefore S = 0, 2$

$\therefore$ degeneracy corresponding $S = 0,\ 2S + 1 = 1$ and

degeneracy corresponding $S = 2,\ 2S + 1 = 2 \times 2 + 1 = 5$

$\therefore$ Total degeneracy = 6

30. $$\Delta A \cdot \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$$

$$\therefore (\Delta L_x)(\Delta L_y) \geq \frac{1}{2} \left| \langle [L_x, L_y] \rangle \right|$$

$$\geq \frac{\hbar}{2} \left| \langle L_z \rangle \right|$$

$$\geq \frac{m\hbar^2}{2}$$

31. (4)
32. (2)
33. (1)
34. (2)

35. Probability $=\dfrac{2N!}{\left(2^{N}N!\right)^{2}}$ for N-step

P = required probability $=\dfrac{12!}{\left(2^{6}\times 6!\right)^{2}}$

$P=0.226$

36. $n=3,\ g=5$

For classical particle

$r_{clas}=g_i^{\,ni}=5^3=125$

For fermions

$r_f={}^{g_i}C_{n_i}={}^{5}C_3=\dfrac{5!}{3!2!}=10$

For Bosons

$r_B={}^{g_i+n_i-1}C_{n_i}$

$={}^{5+3-1}C_3$

$={}^{7}C_3$

$=\dfrac{7!}{4!3!}=35$

$r_c:r_F:r_B=125:10:35\Rightarrow 25:2:7$

37. $\omega=\int\rho\, dV$

$V=\dfrac{N}{\rho}$

$dV=-\dfrac{N}{\rho^2}d\rho$

$\omega=-N\int_{\rho_0}^{2\rho_0}\dfrac{1}{\rho^2}\left(a\rho+b\rho^2\right)d\rho$

$=-N\int_{\rho_0}^{2\rho_0}\left(\dfrac{a}{\rho}+b\right)d\rho$

$=-N\left[a\ln\rho+b\rho\right]_{\rho_0}^{2\rho_0}$

$=-N\left[a\ln 2\rho_0+2b\rho_0-a\ln\rho_0-b\rho_0\right]$

$=-N\left[a\ln\left(\dfrac{2\rho_0}{\rho_0}\right)+b\rho_0\right]$

$=-N\left[a\ln 2+b\rho_0\right]$

$\omega=-\rho_0V_0\left[a\ln 2+b\rho_0\right]$

38. (3)

39. (3)

40. (1)

41. Given: $v_s=3\times10^3 ms^{-1},\ a=3\times10^{-10}m$

linear monoatomic lattice, $\omega_{\max}=?$

In a linear monoatomic lattice, under the long wavelength limit $k\to 0$, the velocity of sound is given by

$$v_s=\frac{\omega_m a}{2}$$

or $$\omega_m=\frac{2v_s}{a}$$

$=\dfrac{2\times3\times10^3}{3\times10^{-10}}$

$=2\times10^{13}\text{Hz}$

42. (2)

43. Given: structure is *fcc* so that

$n=4, r=14\,\text{Å}=1.4\times10^{-10}m,$

$E_f=(at\, 0K)=?$

We know that the relationship between lattice parameter and radius of the atom for *fcc* structure is

$$\sqrt{2}a=4r$$

or $$a=\frac{4r}{\sqrt{2}}$$

$=\dfrac{4\times1.4\times10^{-10}}{\sqrt{2}}$

$=3.96\times10^{-10}m$

Now, the number of electron per unit volume in silver is

$$n'=\frac{n}{a^3}=\frac{4}{\left(3.96\times10^{-10}\right)^3}=6.44\times10^{28}/m^3$$

Thus, the fermi energy at absolute zero is

$E_F=3.65\times10^{-19}\left(n'\right)^{2/3}$

$=3.65\times10^{-19}\left(64.4\times10^{27}\right)^{2/3}$

$=5.86eV$

$=5.86\times1.6\times10^{-19}J$

$=9.38\times10^{-19}J$

44. $A^2=\left(\dfrac{2b}{\pi}\right)^{1/2}$

$\langle T\rangle=\dfrac{\hbar^2}{2m}$

$$\langle V\rangle = A^2\alpha\int_{-\infty}^{\infty}|x|e^{-2bx^2}dx$$

$$= 2\alpha A^2\int_0^{\infty}xe^{-2bx^2}$$

$$= 2\alpha A^2\frac{1}{2}\frac{\sqrt{\frac{1+1}{2}}}{(2b)^{\frac{1+1}{2}}}$$

$$= 2\alpha A^2\frac{1}{2}\frac{\sqrt{1}}{(2b)}$$

$$= \alpha\left(\frac{2b}{\pi}\right)^{1/2}\left(\frac{1}{2b}\right)$$

$$= \alpha\sqrt{\frac{2b}{4b^2\pi}}$$

$$\langle V\rangle = \alpha\sqrt{\frac{1}{2\pi b}} = \frac{\alpha}{\sqrt{2\pi b}}$$

$$\langle H\rangle = \langle T\rangle + \langle V\rangle$$

$$= \frac{\hbar^2 b}{2m} + \frac{\alpha}{\sqrt{2\pi b}}$$

$$\frac{\partial H}{\partial b} = \frac{\hbar^2}{2m} - \frac{1}{2}\frac{\alpha}{\sqrt{2\pi}}b^{-3/2} = 0$$

$$b^{-3/2} = \frac{\alpha}{\sqrt{2\pi}}\frac{m}{\hbar^2}$$

$$\boxed{b = \left(\frac{m\alpha}{\sqrt{2\pi}\hbar^2}\right)^{2/3}}$$

$$\langle H\rangle_{\min} = \frac{\hbar^2}{2m}\left(\frac{m\alpha}{\sqrt{2\pi}\hbar^2}\right)^{2/3} + \frac{\alpha}{\sqrt{2\pi}}\left(\frac{\sqrt{2\pi}\hbar^2}{m\alpha}\right)^{1/3}$$

$$= \frac{\alpha^{2/3}.\hbar^{2/3}}{m^{1/3}(2\pi)^{1/3}}\left(\frac{1}{2}+1\right)$$

$$H_{\min} = \frac{3}{2}\left(\frac{\alpha^2\hbar^2}{2\pi m}\right)^{1/3}$$

45. (3)

46. Since $E_m^{(0)} = \frac{m^2\hbar^2}{2I}$ and corresponding wave function $\Psi_m = \frac{1}{\sqrt{2\pi}}e^{im\phi}$ for ground state $m=0$

$$\Psi = \frac{1}{\sqrt{2\pi}}$$

$$E_{G.S}^{(1)} = K\int_0^{2\pi}\Psi^*\cos^2\phi\,\Psi\,d\phi$$

$$= K\int_0^{2\pi}\frac{1}{2\pi}\cos^2\phi\,d\phi$$

$$= \frac{K}{2\pi}\int_0^{2\pi}\cos^2\phi\,d\phi$$

$$= \frac{K}{2\pi}\times 2\pi\times\frac{1}{2} = \frac{K}{2}$$

$$\boxed{E_{G.S}^{(1)} = \frac{K}{2}}$$

47.

$$\frac{d^2\phi}{dx^2} + \frac{8mE}{\hbar^2}\Psi = 0$$

$$m^2 + K^2 = 0$$

$$m^2 = -K^2$$

$$\boxed{m = \pm iK}$$

$$\Psi = Ae^{iKx} + Be^{-iKr}$$

$$\Psi = A\cos Kr + B\sin Kr$$

$$\Psi(r=0) = 0 = A \Rightarrow A = 0$$

$$\Psi = B\sin Kr$$

$$\Psi(r=a) = B\sin Ka \Rightarrow \sin Ka = 0$$

$$\sin Ka = \sin n\pi \Rightarrow K = \frac{n\pi}{a}, K^2 = \frac{n^2\pi^2}{a^2}$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2} \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$E_2 = \frac{4\pi^2\hbar^2}{2ma^2} = 2\pi^2\hbar^2/ma^2$$

48. K.E. = $E - m_0c^2$ = Rest mass energy

$$E - m_0c^2 = m_0c^2$$

$$E = 2m_0c^2$$

$$\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}c^2 = 2m_0c^2$$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2}$$

$$v=\frac{\sqrt{3}}{2}c$$

49. (2)
50. (1)
51. (3)
52. (4)
53. (1)
54. (2)
55. (2)
56. (1)

57. Given: $v=3\times10^{3}m/s, a=3\times10^{-10}m$

Cut-off frequency $\omega_{\max}=?$

As we known that the cut off frequency occurs at $k=\pi/a$. Further, the velocity and frequency are related through the equation

$$\omega=vk$$

or $$\omega_{\max}=\frac{v\times\pi}{a}=\frac{3\times10^{3}\times3.14}{3\times10^{-10}}$$

$$=3.14\times10^{-10}\text{ rad/sec}$$

58. (2)
59. (3)
60. (2)

61. $$S=NK_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)$$

$$TdS=dE+PdV-\mu dN$$

$$\mu=-\frac{TdS}{dN}\bigg|_{E,V}$$

$$\mu=-T\frac{\partial}{\partial N}NK_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)$$

$$=-K_BT\left\{N\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)\right\}$$

$$=-K_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)+N\frac{\partial}{\partial N}\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)\right]$$

$$=-K_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)+N\frac{\partial}{\partial N}\left(aVE^{3/2}-N^{5/2}\right)\right]$$

$$=-K_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-N\frac{\partial}{\partial N}\left(\ln N^{5/2}\right)\right]$$

$$=-K_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-N\frac{5}{2}\frac{N^{3/2}}{N^{5/2}}\right]$$

$$\mu=-K_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-\frac{5}{2}\frac{N^{5/2}}{N^{5/2}}\right]$$

$$\mu=-K_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-\frac{5}{2}\right]$$

62. No of ways in which sum is 15 is given by

$(5,5,5)(4,5,6)(6,5,4)(5,4,6)(5,6,4)(4,6,5)$
$(6,4,5)(3,6,6)(6,3,6)(6,6,3)$

total ways = 10

Dice can be thrown in $6\times6\times6=216$ ways

$\therefore$ Probability $P=\frac{10}{216}=\frac{5}{108}$

63. $$E=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+\frac{p_z^2}{2m}+\frac{1}{2}K\left(x^2+y^2+z^2\right)$$

$$E=\frac{1}{2}K_BT+\frac{1}{2}K_BT+\frac{1}{2}K_BT$$
$$+\frac{1}{2}K_BT+\frac{1}{2}K_BT+\frac{1}{2}K_BT$$
$$=\frac{6}{2}K_BT$$
$$=3K_BT$$

For N particles

$E=3NK_BT$

64. $$V(r,\theta)=\left(a_lr^l+\frac{b}{r^{l+1}}\right)P_l^m(\cos\theta)$$

$P_l^m(\cos\theta)=1$ if $l=0$

$P_l^m(\cos\theta)=\cos\theta$ if $l=1$

Given, $V(r,\theta,\phi)=f(r)\cos\theta$

$$V(r,\theta)=ar+\frac{b}{r^2}$$

65. For melting of ice into water at constant pressure, the Gibbs energy does not exhibit a discontinuous changing across the phase transition.

66. The Fourier transform of the derivative of the

$$F\{\delta'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \delta'(x)\, dx$$

$$F\{\delta'(x)\} = \int_{-\infty}^{\infty} e^{-iKx} \delta'(x)\, dx$$

$$= e^{-iKx} \int_{-\infty}^{\infty} \delta'(x)\, dx - (-iK) \int_{-\infty}^{\infty} e^{-iKx} \delta(x)\, dx$$

$$= iK$$

Using the property $\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0)$

and $\int_{-\infty}^{\infty} \delta'(x)\, dx = 0$

67. $\dot{x}(t) = y(t) \Rightarrow \ddot{x} = \dot{y}(t) \Rightarrow \ddot{x} = -x(t)$

$\dot{y} = -x(t) \Rightarrow \ddot{y} = -\dot{x}(t) \Rightarrow y = -y(t)$

$\vec{F} = \ddot{x}\hat{i} + \ddot{y}\hat{j} = -(x\hat{i} + y\hat{j})$

$\frac{dV}{dx} = x, \quad \frac{dV}{dy} = y$

$V = \frac{x^2}{2}, \quad V = \frac{y^2}{2}$

Therefore potential, $V = \frac{1}{2}(x^2 + y^2)$

68. If

$F(x,y) = x^a y^b$ or $F(x,y) = \frac{x^a}{y^b}$

Then $\frac{\Delta F}{F} = \sqrt{\left(a\frac{\Delta x}{a}\right)^2 + \left(b\frac{\Delta y}{y}\right)^2}$

$= \sqrt{1^2 + (4\times 3)^2}$

$= \sqrt{1 + 12^2}$

$= 12\%$

69. -

70. (4)

71. (1)

72. (1)

73. (1)

74. Given:

$M_1 = 202, T_{C1} = 4.159K,$

$\alpha = 0.5, M_2 = 200.7, T_{C2} = ?$

According to isotope effect, we known that

$T_c \alpha M^{-\alpha}$

or $M^{\alpha} T_c = \text{Constant}$

therefore, we can write

$(202)^{1/2} \times 4.159 = (200.7)^{1/2} \times T_{c2}$

or $T_{c2} = \left(\frac{202}{200.7}\right)^{1/2} \times 4.159$

= 4.172 K

75. Given:

$\lambda(3K) = 396\,\mathring{A},$

$\lambda(7.1K) = 1730\,\mathring{A}, T_c = ?$

In a superconductor, the penetration depth is given by

$$\lambda(T) = \lambda(0)\left[1 - \left(\frac{T}{T_c}\right)^4\right]^{-1/2}$$

So that $\lambda(3) = \lambda(0)\left[1 - \left(\frac{3}{T_c}\right)^4\right]^{-1/2}$ ….(1)

and $\lambda(7.1) = \lambda(0)\left[1 - \left(\frac{7.1}{T_c}\right)^4\right]^{-1/2}$ …..(2)

Removing the power $(-1/2)$ and taking the ratio, we obtain

$$\left[\frac{\lambda(7.1)}{\lambda(3)}\right]^3 = \frac{T_c^4 - (3)^4}{T_c^4 - (7.1)^4}$$

or $\left(\frac{1730}{396}\right)^2 = \frac{T_c^4 - 81}{T_c^4 - 2541}$

or $19 = \frac{T_c^4 - 81}{T_c^4 - 2541}$

or $T_c^4 = \frac{19 \times 2541 - 81}{81}$

or $T_c = 7.193K$