1. **Solution.** (b)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ 0 & -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$$

$$\Rightarrow \quad A^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{n\pi}{3} & \sin\frac{n\pi}{3} \\ 0 & \sin\frac{n\pi}{3} & \cos\frac{n\pi}{3} \end{pmatrix}$$

$\Rightarrow \quad A^6 = I$

2. **Solution.** (d)

$$J = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}_{101}$$

J has 1 eigenvalue = 101 and remaining 100 eigen values = 0

So, J-I has 1 eigenvalue 101 – 1 = 100

and remaining 100 eigenvalues as 0 – 1 = -1

So, $|\boldsymbol{J} - \boldsymbol{I}| = 100 \times (-1)^{100} = 100$

3. **Solution.** (b)

$$W(x)_{x=0} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$W(x)_{x=1} = \int_0^1 \frac{P_1}{P_0} dx$$

$$W(x)_{x=0} = (-1)_e \int_0^1 2x dx = -e^{x^2}\Big|_0^1 = -e$$

4. Solution. (a,c)

$$x_{n+1} = f(x_n)\sqrt{2+x_n}$$

$$\lim_{n\to\infty} x_n = l \text{ (Let)}$$

Then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\sqrt{2+x_n}$

$\Rightarrow \quad l = \sqrt{2+l}$

$\Rightarrow \quad l^2 - l - 2 = 0$

$\Rightarrow \quad (l-2)(l+1) = 0$

$\Rightarrow \quad l = 2 - 1$

As $x_0 = 1,\ x_1 = \sqrt{2+1}, \dots$

So, sequence is monotonically increasing and positive,

So, $l = 2$.

Also, $\sqrt{2+\sqrt{2+\sqrt{2+\dots}}} = 2$

5. Solution. (d)

$$A = I - \begin{bmatrix} 1/n & 1/n & \dots & 1/n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1/n & 1/n & \dots & 1/n \end{bmatrix}$$

eigen values of A is 0, 1 times and is equal to 1; n – 1 times.

6. Solution. (a)

Ax = 0 has trivial solution only so Rank (A) = 4.

7. Solution. (b,c)

$$y(t) = 1 - y^2(t)$$

$\Rightarrow \quad \dfrac{dy}{1-y^2} = dt$

$\Rightarrow \quad \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-y}\right) = t + c$

$y(0) = 0$

$\Rightarrow \quad \dfrac{1}{2}\ln(1) = 0 + c \qquad \Rightarrow c = 0$

$$\Rightarrow \ln\left(\frac{1+y}{1-y}\right) = 2$$

y = –1 (for log to be defined)

Also, $\frac{1+y}{1-y} = e^{2t} \qquad \Rightarrow \frac{1+y}{2} = \frac{e^{2t}}{1+e^{2t}}$

$$\Rightarrow y = \frac{2e^{2t}}{1+e^{2t}}$$

Now, $\frac{dy}{dt} > 0 \ \forall \ t \in R$,

So, y(t) is strictly increasing in **R**.

8. Solution. (d)

For small x, sinx = x.

So, the equation given by

$\frac{d^2x}{dt^2} = -\sin x$ can be approximated as $\frac{d^2x}{dt^2} = -x$

$\Rightarrow \qquad (D^2+1)x = 0$

$\Rightarrow \qquad x(t) = a\cos t + b\sin t$

9. Solution. (b)

By rational in clockwise direction through 90^0

$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = (0)\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = (1)\begin{bmatrix}1\\0\end{bmatrix} + (0)\begin{bmatrix}0\\1\end{bmatrix}$$

With respect to the standard ordered basis $\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$, transformation matrix is $T = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$.

10. Solution. (c)

$a_n = z^{n!} \qquad \Rightarrow (a_n)^{1/n} = z^{(n-1)!}$

$\lim(a_n)^{1/n} = \lim Z^{(n-1)!} = 0 < 1$

if $|z| < 1$ and does not exists if $|z| > 1$.

So, radius of convergence is 1.

11. Solution. (a)

(D – 1)y = 17

$\Rightarrow \qquad$ C.F. is y = ce^x $\qquad$ P.I. is $\frac{1}{D-1}(17) = -17$

So, general solution is $y = ce^{x} - 17$

$y_1(0) = 0 \quad \Rightarrow 0 = c - 17 \quad \Rightarrow c = 17 \quad \Rightarrow y_1 = 17e^x - 17$

$y_2(0) = 1 \quad \Rightarrow 1 = e - 17 \quad \Rightarrow c = 18 \quad \Rightarrow y_2 = 18e^x - 17$

y_1 and y_2 are monotonically increasing and at x = 0 if their value is different, then they will never intersect.

12. Solution. (d)

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6} \text{ (given)}$$

$$\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)+\frac{1}{4}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right)=\frac{\pi^2}{6}$$

$$\Rightarrow \quad \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)+\frac{1}{4}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{6}$$

$$\Rightarrow \quad \frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots=\frac{\pi^2}{8}-1$$

13. Solution. (c)

$$\sum_{n=0}^{\infty} 3^{-n}(z-1)^2 \text{ converges if } \lim\left|3^{-n}(z-1)^{2n}\right|^{1/n}<1$$

$$\Rightarrow \quad \left|\frac{(z-1)^2}{3}\right|<1 \quad \Rightarrow |z-1|^2<3$$

$$\Rightarrow |z-1|<\sqrt{3}$$

14. Solution. (c)

$|A| = 0$

$\Rightarrow$ one eigenvalues is 0 if $\lambda = 0$ is another (second) then third eigenvalues is

trace (A) - λ = 20 - λ

15. Solution. (c)

$A^2 = 0 \Rightarrow$ A is nilpotent

$\Rightarrow$ A is not diagonalizable

$\Rightarrow$ A has only 0 as eigenvalues with multiplicity 2

$Av = v \quad \Rightarrow (A - I)v = 0 \quad \Rightarrow v = 0$

as, $|A - I = (0-1)(0-1) = 1$

So, A – I is not singular.

16. Solution.(4)

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos x dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \left(2\cos^2 x - 1\right)\sin x dx$$

$$= -\frac{1}{\pi}\int_{1}^{-1} \left(2t^2 - 1\right)dt \qquad (\cos x = t)$$

$$= -\frac{2}{3\pi}$$

17. (2)

Equation of surface is

$$z^2 = \frac{3}{2}\left(x^2 + y^2\right)$$

$$F = 3x^2 + 3y^2 - 2z^2 = 0$$

Unit normal to the surface

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{6x\hat{i} + 6y\hat{j} - 4z\hat{k}}{2\sqrt{9x^2 + 9y^2 + 4z^2}}$$

$$= \frac{3x\hat{i} + 3y\hat{j} - 2z\hat{k}}{\sqrt{9x^2 + 9y^2 + 4z^2}}$$

$$= \frac{3.\sqrt{\frac{2}{3}}\hat{i} - 2\hat{k}}{\sqrt{9 \times \frac{2}{3} + 4}} = \frac{\sqrt{6}\hat{i} - 2\hat{k}}{\sqrt{10}}$$

$$= \sqrt{\frac{3}{5}}\hat{i} - \frac{2}{\sqrt{10}}\hat{k}$$

18. Solution.(1)

$$\frac{d^2y}{dx^2} - y = 0$$

$$\left(D^2 - 1\right)y = 0$$

$$y = c_1 e^{x} + c_2 e^{-x}$$

$$y(0) = 0 \Rightarrow c_1 + c_2 = 0$$

$$y(-\infty) = 0$$

So, $c_1 = c_2 = 0$

Has no solution

19. Solution: (2)

$$f(z)=(1+z)\left(1+z^3\right)\left(1+z^4\right).......\left(1+z^{3m}\right)$$

$$\log f(z)=\log(1+z)+\log\left(1+z^3\right)+\log\left(1+z^6\right)+......\left(1+z^{3m}\right)$$

$$\frac{1}{f(z)}f'(z)=\frac{1}{1+z}+\frac{3z^2}{1+z^3}+\frac{6z^5}{1+z^6}+.......+\frac{3mz^{3m-1}}{1+z^{3m}}$$

At $z=0$,

$$\frac{1}{f(0)}f'(0)=1$$

$$f'(0)=f(0)=1$$

20. Solution.(3)

σ_i are Pauli matrices

$$[\sigma_i,\sigma_2]=i\sigma_3$$
$$[\sigma_2,\sigma_3]=i\sigma_1$$
$$[\sigma_3,\sigma_1]=i\sigma_2$$
$$[\sigma_i,\sigma_i]=0$$

$$[\sigma_1,[\sigma_2,\sigma_3]]+[\sigma_2,[\sigma_3,\sigma_1]]+[\sigma_3,[\sigma_1,\sigma_2]]$$
$$=[\sigma_1,i\sigma_i]+[\sigma_2,i\sigma_2]+[\sigma_3,i\sigma_3]$$
$$=[\sigma_1,\sigma_1]+i[\sigma_2,\sigma_2]+i[\sigma_3,\sigma_3]$$
$$=0$$

21. Solution. (2)

$$P^2=P$$
$$\Rightarrow P^2-P=0$$
$$\Rightarrow P(P-1)=0$$

The eigenvalue can be either 0 or 1. Since, $Tr(P)=n-1$and trace is equal to sum of eigenvalues so, one of the eigenvalues will be 0. Hence, Det(P) = product of eigenvalues = 0

22. Solution.(2)

$$r\sin\left(\theta+\frac{\pi}{4}\right)=5$$

$$r\sin\theta\cos\frac{\pi}{4}+r\cos\theta\sin\frac{\pi}{4}=5$$

$x+y=5\sqrt{2}$ (straight line)

23. Solution. (3)

$$\frac{d^2y}{dt^2}+y=\sin t$$

$$\left(D^2+1\right)y=\sin t$$

$$C.F.=A\sin t+B\cos t$$

$$P.I. = \frac{1}{D^2+1}\sin t = -\frac{t}{2}\cos t$$

$$y = C.F. + P.I$$

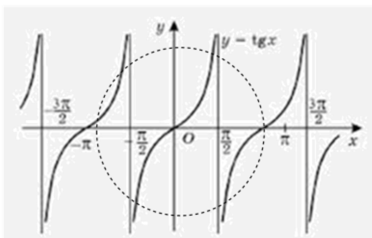
$$= A\sin t + B\cos t - \frac{t}{2}\cos t$$

$y(0)=0 \Rightarrow B=0$

$y(t) = A\sin t - \frac{t}{2}\cos t$

Where A is arbitrary constant.

24. Solution.(3)



25. Solution.(1)

There are two possibilities to get sum 28 of the dots on the faces of five dice either 4-dice has 6-dots and 1-has four dots or 3-dice has 6-dots and 2-dice has five dots. So the probabilities is

$$\frac{{}^5C_1 + {}^5C_2}{6^5} = \frac{16}{7776}$$

26. Solution(2)

$$f(x) = f(\log\pi/2) + (x-\log\pi/2)f'(\log\pi/2) + \frac{(x-\log\pi/2)^2 f''(\log\pi/2)}{2!} + \ldots\ldots$$

Where $f(\log\pi/2)=1$

$f'(\log\pi/2)=0$

$f''(\log\pi/2) = -(\pi/2)^2$

$\therefore$ $\boxed{f(x) = 1 - \frac{\pi^2}{8}\left[x-\log(\pi/2)\right]^2 + \ldots\ldots}$

27. Solution. (2)

Equation of tangent plane to the surface

$f(x,y,z)=c$, where c is constant at the point $P(x_0,y_0,z_0)$ is

$$(x-x_0)f_x(x_0,y_0,z_0) + (y-y_0)f_y(x_0,y_0,z_0) + (z-z_0)f_z(x_0,y_0,z_0) = 0$$

$f(x,y,z) = x^2+y^2-z$

$\frac{\partial f(x,y,z)}{\partial x} = 2x$, $\frac{\partial f(x,y,z)}{\partial y} = 2y$, $\frac{\partial f(x,y,z)}{\partial z} = -1$

So, $2(x-1)+4(y-2)-(z-3)=0$

$\Rightarrow 2x+4y-z=7$ is equation of tangent plane to the given surface

28. Solution. (3)

29. Solution.(3)

Given that , $A^n = I$.

The product of the eigenvalue of matrix A is,

$e^{i\pi/5}.e^{i\pi/6} = e^{11\pi i/30}$

$\left(e^{\frac{11\pi i}{30}}\right)^n = e^{2\pi i m}$; where, m is an integer

Or $\frac{11ni\pi}{30} = 2\pi im$, so, $n = \frac{30}{11\pi i} \times 2\pi i \times m = \frac{60m}{11}$

So, the smallest value of n such that $m = 11$ is, $n = \frac{60}{11} \times 11 = 60$

30. Solution.(3)

31. Solution. (4)

On mixing, temperature of gaseous mixture will remain T

So, ΔS for each gaseous system

$= \int \frac{dQ}{T}$

$= \int \frac{dW}{T}$

$= \int \frac{PdV}{T}$

$= \int_V^{2V} \frac{R}{V} dV$

Total entropy change in gaseous system

$\Delta S = 2R \ln 2$

$= \frac{2PV}{T} \ln 2$

32. Solution.(1)

According to Wien's Displacement law

$\lambda_m T = const$

$\Rightarrow \lambda_m \propto \frac{1}{T}$

Or $v_{\max} \propto T$

So, if $v_{\max}$ becomes half, T will also becomes half.

$T_2 \frac{T_1}{2}$

33. Solution.(1)

RMS velocities of the molecules are

$$V_1 = \sqrt{\frac{8kT}{\pi m}}$$

$$V_1 = \sqrt{\frac{8kT}{4\pi m}} = \frac{1}{2}\sqrt{\frac{8kT}{\pi m}}$$

Root mean square velocity of atoms in the mixture

$$V = \frac{V_1 + V_2}{2} = \frac{3}{4}\sqrt{\frac{8kT}{\pi m}}$$ [Since the number of molecules are equal]

34. Solution.(1)

Two solids at temperature T_1 & T_2 are brought in contact

$\because T_1 > T_2$, the solid at T_1 will loose heat to the solid at T_2

Let equilibrium temperature be T

Heat loss by solid at T_1 = Heat gained by solid at T_2

$$\Rightarrow C(T_1 - T) = C(T - T_2)$$

$$T = \frac{T_1 + T_2}{2}$$

$\Delta S_1 =$ Change in entropy of solid initially at T_1

$$= \int \frac{dQ}{T} = C\int_{T_1}^{T} \frac{dT}{T} = C\ln\frac{T}{T_1}$$

$\Delta S_1 =$ Change in entropy of solid initially at T_2

$$\int \frac{dQ}{T} = C\int_{T_2}^{T} \frac{dT}{T} = C\ln\frac{T}{T_2}$$

There is no change in entropy of surrounding change in entropy of universe = change in entropy of system + change in entropy of surrounding

$$\Delta S = \Delta S_1 + \Delta S_2$$

$$= C\ln\frac{T}{T_1} + C\ln\frac{T}{T_2}$$

$$= C\ln\frac{T^2}{T_1 T_2}$$

$$= C\ln\left[\frac{(T_1 + T_2)^2}{4T_1 T_2}\right]$$

35. Solution. (3)

Bose Einstein distribution function is given by

$f(E) = \frac{1}{e^{(E-\mu)\beta} - 1}$, where μ is chemical potential and $\beta = \frac{1}{k_B T}$. In Bose Einstein condensation all the Bose particles occupy the ground state. So $f(E) = 1$

Here, $f(E)$ denotes the probability of occupancy of particle which lies in certain state, Now Bose Einstein distribution function becomes

$$1 = \frac{1}{e^{(\in-\mu)/k_BT} - 1}$$

$$e^{(\in-\mu)/k_BT} - 1 = 1$$

$$e^{(\in-\mu)/k_BT} = 2$$

$$\log e^{(\in-\mu)/k_BT} = \log 2$$

$$\frac{\in-\mu}{k_BT} = \log 2$$

$$(\in-\mu) = k_BT \log \quad \text{.............}(1)$$

For Bose Einstein condensation occur, $T \to 0$ then, equation (1) becomes, $\in-\mu = 0, or\ \mu = \in$

In 3-D, Harmonic oscillator $E_n = (n + 3/2)\hbar\omega$ for ground state, $n = 0$

Chemical potential is $\mu = (n + 3/2)\hbar\omega$ $\boxed{\mu = 3/2\hbar\omega}$

36. Solution.(2)

37. Solution.(2)

Bose Einstein temperature is given by

$$T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612V}\right)^{2/3} \text{ or } T_B \alpha \left(\frac{N}{V}\right)^{2/3}$$

So $T_B \alpha n^{2/3}$

Where concentration of particle, $n = N/V$. Let T_{B_1} is temperature at which Bose condensation occurs in liquid H_e^4 and T_{B_2} for H_e^4 in gaseous state.

$$T_{B_1} \alpha (n_1)^{2/3} \quad \text{............}(1)$$

$$T_{B_2} \alpha (n_2)^{2/3} \quad \text{.............}(2)$$

Dividing (1) by (2), we get

$$\frac{T_{B_1}}{T_{B_2}} = \left(\frac{n_1}{n_2}\right)^{2/3}, \text{where } T_{B_1} = 2.17K, \frac{n_1}{n_2} = 1000$$

$$\frac{2.17K}{T_{B_2}} = (1000)^{2/3}, \text{ or } \boxed{T_{B_2} = 21.7mk}$$

38. Solution.(3)

For Black body radiation

$E = \sigma T^4$; where σ is the Stefan's constant

The entropy of the radiation is

$$S_T = \int \frac{\partial E}{T} = \int \frac{\partial(\sigma T^4)}{T} = 4\sigma \int \frac{T^3}{T} dT$$

$$S_T \alpha T^3$$

$$\frac{S_T}{S_{2T}} = \frac{T^3}{8T^3} \quad \boxed{S_{2T} = 8S_T}$$

39. (2),	40.(2),	41.(2),	42.(3),	43.(2),	44.(2),	45.(3)
46. (3),	47. --	48.(4),	49.(1),	50.(3)		